Journal of Statistical Physics, Vol. 48, Nos. 1/2, 1987

On the Large-Coupling-Constant Behavior of the Liapunov Exponent in a Binary Alloy

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Received October 15, 1986

We consider the usual one-dimensional tight-binding Anderson model with the random potential taking only two values, 0 and λ, with probability p and $1-p$, $0<p<1$. We show that the Liapunov exponent $\gamma_\lambda(E)$, $E\in\mathbf{R}$, diverges as $\lambda\to\infty$ uniformly in the energy E. Using a result of Carmona, Klein, and Martinelli, this proves that for λ large enough, the integrated density of states is singular continuous. We also compute explicitly the exact asymptotics for a dense set of energies and we compare the results with numerical simulations.

KEY WORDS: Liapunov exponent; random matrices; one-dimensional Anderson localization.

1. INTRODUCTION

The one-dimensional discrete Schrödinger equation with a random stationary and ergodic potential has attracted the attention of both physicists and mathematicians for many years.[(1-3)] In particular, for a large class of models it was proved that if the potential v is a collection of independent, identically distributed random variables, then, under very mild assumptions on the probability distribution of the potential $dP(v)$, the eigenfunctions of the infinite Jacobi matrix H,

$$(H\varphi)(n) = -\varphi(n+1) - \varphi(n-1) + v(n)\,\varphi(n), \qquad n\in\mathbf{Z} \tag{1.1}$$

are exponentially localized with probability one whenever $\lambda>0$.[(1-3)]

The rate of decay ("mass") at energy E is given by the so-called "Liapunov exponent" $\gamma(E)$, which can be defined as follows. Let $T_n(v, E)$ be the 2×2 matrix

$$T_n(v, E) = \begin{pmatrix} v(n)-E, & -1 \\ 1, & 0 \end{pmatrix} \tag{1.2}$$

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0022-4715/87/0700-0001$05.00/0

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Then

$$\lim_{n \to \infty} n^{-1} \log \|T_n(v, E) \cdots T_1(v, E)\| = \gamma(E) \tag{1.3}$$

exists with probability one, it is a *nonrandom* quantity, and it is *strictly positive* for any $E \in \mathbf{R}$.

Surprisingly, a very natural model in which the random variables are Bernoulli-distributed, e.g.,

$$\begin{aligned} v &= 0 \qquad \text{with probability } p \\ v &= \lambda \qquad \text{with probability } 1-p \end{aligned} \tag{1.4}$$

turned out to be very difficult to analyze, and only recently have Carmona *et al.*[4] been able to prove the localization phenomena in this case. An interesting consequence of their result was a very direct connection between the regularity properties of the integrated density of states (i.d.s.) measure $dN_\lambda(E)$ and the size of the Liapunov exponent. The i.d.s. is defined as follows: Let H_L be the restriction of the Jacobi matrix H to the interval $[-L, +L]$. Then

$$N_\lambda(E) = \lim_{L \to \infty} (2L+1)^{-1} \#\{\text{eigenvalues of } H_L < E\} \tag{1.5}$$

The above limit exists almost surely and it is a nondecreasing function of E.

Carmona *et al.*[4] proved the following result. Let Σ be the almost surely constant spectrum of H; in the Bernoulli case, $\Sigma = [-2, +2] \cup [\lambda - 2, \lambda + 2]$. Then we have the following result.

Theorem. For any open set $I \subset \Sigma$ such that

$$\inf_{E \in I} \gamma_\lambda(E) > \ln 4 \tag{1.6}$$

the restriction of the measure dN_λ to I is purely singular continuous.

In a slightly less precise form this result was conjectured in Ref. 5. At first sight it is natural to conjecture that the critical condition (1.6) is satisfied for $I = \Sigma$ provided that the coupling constant λ is taken large enough. In fact, if one expresses the Liapunov exponent by means of the Thouless formula[6]

$$\gamma_\lambda(E) = \int_{-2}^{2} dN_\lambda(E') \ln |E - E'| + \int_{\lambda-2}^{\lambda+2} dN_\lambda(E') \ln |E - E'| \tag{1.7}$$

and E is, e.g., in the lowest band $[-2, 2]$, then the second term in the rhs of (1.7) behaves like

$$\int_{\lambda-2}^{\lambda+2} dN_\lambda(E') \ln |E-E'| \approx \ln \lambda \int_{\lambda-2}^{\lambda+2} dN_\lambda(E') = (1-p) \ln \lambda \qquad (1.8)$$

as $\lambda \to \infty$.

This would be exactly the naive conjecture for the large-λ behavior of $\gamma_\lambda(E)$: the $\ln \lambda$ factor is the contribution to the Liapunov exponent of a single high barrier $v=\lambda$ and $(1-p)$ is the density of the high barriers. However, the first term in (1.7) contains a *negative singularity* at $E'=E$ which could *a priori* compensate the large positive term (1.8). It is clear that the regularity properties of the function $N_\lambda(E)$ must play an important role. In general, it can be proved[7] that if the single-site distribution has a bounded density, then the $dN_\lambda(E)$ has also a bounded density, and Le Page[8] (see also Ref. 3) proved that under very mild conditions, which cover the Bernoulli case, $N_\lambda(E)$ is at least Holder continuous. However, in the Bernoulli case, Simon and Taylor,[5] using an argument due to Halperin,[9] showed that the order $\alpha=\alpha(\lambda)$ of Holder continuity of $N_\lambda(E)$ decreases as $\lambda \to +\infty$ like

$$\alpha(\lambda) = O(1/\lambda) \qquad (1.9)$$

at least for a dense set of energies.

Nieuwenhuizen and Luck, in a very interesting, although not completely rigorous paper,[10] analyzed in great detail the nature of the integrated density of states near these special energies and derived, e.g., the following result:

$$N_\lambda(E \pm \varepsilon) - N_\lambda(E) \approx \pm \varepsilon^{2\alpha} R_\pm(\ln \varepsilon / \ln \mu)$$

where E is one of the special energies, $R_\pm$ is a periodic function, and α and μ depend on E.

The above results show that the first term in (1.7) behaves for a dense set of energies like

$$\int_{-2}^{2} dN_\lambda(E') \ln |E-E'| \approx -C(E, p) \ln \lambda \qquad (1.10)$$

for some constant $C(E, p)$.

Thus, one has to show that $C(E, p)$ is always less than $(1-p)$. This turns out to be a very subtle problem, and in Ref. 4 it was partially solved by proving that for $\delta > 0$ there exists a set $\Sigma_0 \subset \Sigma$ of full Lebesgue measure 1 such that

$$\lim_{\lambda \to \infty} \gamma_\lambda(E) > (1-p) - \delta \qquad \text{for any} \quad E \in \Sigma_0 \qquad (1.11)$$

This result, however, did not rule out the possibility that

$$\inf_{E \in \Sigma} \gamma_\lambda(E) < \ln 4 \qquad \text{for any } \lambda \tag{1.12}$$

In this paper we completely solve the problem. We prove that uniformly in the energy the Liapunov exponent goes to infinity like const $\cdot \ln \lambda$ and we compute exactly the constant for a dense set of energies. This is done in Section 3. In Section 2 we recall some mathematical definitions and results concerning the Liapunov exponent, while in Section 4 we present some numerical simulations. The reader interested in the weak disorder case $\lambda \ll 1$ is referred to an interesting paper by Derrida and Gardner[11]; the result of their work indicate that for λ sufficiently small the *density of states* $dN_\lambda(E)/dE$ should exist. Therefore, a transition from weak disorder to high disorder should occur in the regularity properties of the i.d.s.

2. ON THE LIAPUNOV EXPONENT

We recall here a basic mathematical result on the Liapunov exponent that we will need in the next section. The material of this section is taken from Ref. 3.

For any nonzero vector $x \in \mathbf{R}^2$, $\underline{x}$ denotes the corresponding point in the compact projective line X. A matrix $g \in SL(2, \mathbf{R})$ acts on X by

$$g \cdot \underline{x} = g\underline{x} \tag{2.1}$$

Given two probability measures ν and $\tilde{\mu}$ on $SL(2, \mathbf{R})$ and X, respectively, we denote their "convolution" $\nu * \tilde{\mu}$ by

$$(\nu * \tilde{\mu})(f) = \iint f(g \cdot \underline{x})\, d\nu(g)\, d\tilde{\mu}(\underline{x}) \tag{2.2}$$

for any $f \in C_0(X)$.

We will say that $\tilde{\mu}$ is ν-invariant iff $\nu * \tilde{\mu} = \tilde{\mu}$.

For any measure $\tilde{\mu}$ on X such that $\tilde{\mu}\binom{0}{1} = 0$ we will also construct a measure μ on $\mathbf{R}$ via the application $\underline{x} = \binom{u}{v} \to v/u$.

Let now ν_E be the distribution of the transfer matrix $T(v, E)$ given by (1.2) corresponding to the probability distribution $dP(v)$ of the potential v. Then one has the following result.

Theorem. If the measure $dP(v)$ is not concentrated on a single point and if $\int dP(v)\, |v|^\alpha < +\infty$ for some $\alpha > 0$, then:

(i) There exists a unique ν_E-invariant measure $\tilde{\mu}_E$ with $\tilde{\mu}_E(\underline{x}) = 0$ $\forall \underline{x} \in X$.

(ii) For some $\beta > 0$

$$\int_{\mathbf{R}} d\mu_E(z)\,|z|^{\beta} < +\infty; \qquad \int_{\mathbf{R}} d\mu_E(z)\,|z - z_0|^{-\beta} < +\infty \quad \text{for any } z_0 \in \mathbf{R}$$

(iii) The Liapunov exponent $\gamma(E)$ is strictly positive for any E and

$$\gamma(E) = -\int d\mu_E(z) \ln |z|$$

This last formula is known as the Furstemberg formula.

The above result clearly applies to the case considered in this paper, namely

$$dP(v) = p\delta(v = 0) + (1 - p)\,\delta(v = \lambda)$$

In particular, for any energy E, Eq. (2.2) applied to v_E and $\tilde{\mu}_E$ gives the following basic equation satisfied by the measure μ_E:

$$\mu_E(A) = p\mu_E(T_0^{-1}(A)) + (1 - p)\,\mu_E(T_\lambda^{-1}(A)) \tag{2.3}$$

where A is any measurable set in $\mathbf{R}$ and the maps T_0 and T_λ are given by

$$T_0(z) = -1/(E + z); \qquad T_\lambda = 1/(\lambda - E - z) \tag{2.4}$$

In the derivation of (2.3) we have used the regularity properties of the measure μ_E expressed by (ii) of the theorem.

For our purposes it is quite important to observe that for λ large, the map T_λ has two fixed points A and B, one stable and one unstable (see Fig. 1 below).

We will see in the next section that Eq. (2.3) is the basic tool for computing the Liapunov exponent.

3. MAIN RESULTS

In this section we state and prove our main results concerning the asymptotic behavior of the Liapunov exponent $\gamma_\lambda(E)$ as λ goes to infinity and E runs through the spectrum of $\Sigma = [-2, 2] \cup [\lambda - 2, \lambda + 2]$ of H. In the first theorem we show that uniformly in the energy E, $\gamma_\lambda(E)$ goes to infinity as $\lambda \to \infty$, while in the second theorem we compute explicitly the exact rate for a dense set of energies. Since there is a clear symmetry between the lower band $[-2, +2]$ and the upper band $[\lambda - 2, \lambda + 2]$ of Σ, all the results are stated only for $|E| < 2$.

Theorem 1. The following relation holds:

$$\lim_{\lambda \to \infty} \inf_{|E|<2} \gamma_\lambda(E)/\ln \lambda > k(p)$$

with $k(p) > (1-p)^2/2[1+(1-p)p^2]$.

Theorem 2. For any energy E of the form

$$E = 2\cos(\pi k/L+1)$$

where k and $L+1$ are mutually prime integers satisfying $0<k<L+1$, we have

$$\lim_{\lambda \to \infty} \gamma_\lambda(E)/\ln \lambda = 1 - p - (1-p)^2 p^L/(1-p^{L+1})$$

Remark 1. It appears from the result of Theorem 2 that

$$\inf_{|E|<2} \lim_{\lambda \to \infty} \gamma_\lambda(E)/\ln \lambda = \lim_{\lambda \to \infty} \gamma_\lambda(0)/\ln \lambda = (1-p)/(1+p)$$

However, the numerical simulations described in the next section indicate that for large but fixed λ the minimum of $\gamma_\lambda(E)/\ln \lambda$ is attained for $E_0 = -2/\lambda$, where $\gamma_\lambda(E) < (1-p)\ln(\lambda)/2$. This value of the energy is exactly the lowest eigenvalue of the Jacobi matrix $H(v)$ with $v(0)=0$ and $v=\lambda$ elsewhere and, at least to first order in $1/\lambda$, it is also the eigenvalue of *any* finite block of length L containing a site of zero potential surrounded by two sites where $v=\lambda$. Thus, the tunneling at this special energy among these blocks is strongly enhanced, with the result of lowering considerably the Liapunov exponent. That this should be indeed the case is also suggested by the proof of Theorem 1, to which we now turn.

Proof of Theorem 1. The main tool for the proof is a clever use of the invariance of the measure μ_E expressed by formula (2.3). For notational convenience we will suppress the subscript E in the measure μ_E.

We start by integrating by parts in the Furstemberg formula of Section 2. We get

$$\gamma_\lambda(E) = -\int_{-\infty}^{-1} dz\, |z|^{-1}\, \mu(-\infty, z) + \int_0^1 dz\, z^{-1} \mu(0, z) - \int_1^{\infty} dz\, z^{-1} \mu(z, \infty) + \int_{-1}^{0} dz\, |z|^{-1}\, \mu(z, 0) \tag{3.1}$$

Integration by parts holds because of the decay properties of the measure μ discussed in Section 2.

We proceed by showing that the measure μ of the tails $\mu(-\infty, z)$ and $\mu(z, +\infty)$ appearing in the negative terms in (3.1) can be controlled by the measure μ of a neighborhood of the origin. This kind of computation can be easily visualized by looking at Fig. 1.

For $z > 3$ we have

$$\mu(-\infty, -z) = \mu(0, 1/(z-E)) - (1-p)\,\mu(-z, \lambda - z) \tag{3.2}$$

$$\mu(z, +\infty) = \mu(-1/(z+E), 0) + (1-p)\,\mu(z, \lambda + z) \tag{3.3}$$

If $z > \lambda - E$ we also have

$$\mu(z, +\infty) = \mu(-1/(z-(\lambda - E)), 0) - p\mu(z-\lambda, z) \tag{3.4}$$

To prove (3.2) and (3.3), we apply formula (2.3) to the sets $T_0(-\infty, -z)$ and $T_0(z, +\infty)$, respectively. For example, (2.3) applied to $T_0(-\infty, -z)$ gives

$$\begin{aligned}\mu(T_0(-\infty, -z)) &= \mu(0, 1/(z-E))\\ &= p\mu(-\infty, -z) + (1-p)\,\mu(T_\lambda^{-1}(0, 1/(z-E)))\\ &= p\mu(-\infty, -z) + (1-p)\,\mu(-\infty, \lambda - z)\\ &= \mu(-\infty, -z) + (1-p)\,\mu(-z, \lambda - z)\end{aligned} \tag{3.5}$$

and (3.2) follows.

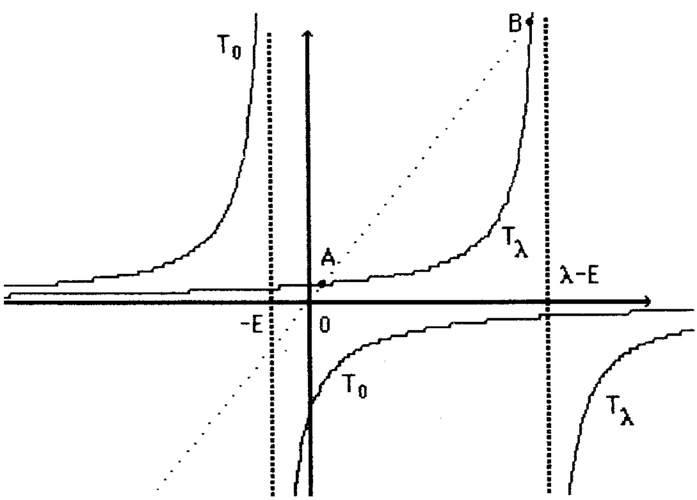

Fig. 1

Expression (3.3) is proved in a similar way. To prove (3.4), we use the set $T_\lambda(z, +\infty)$:

$$\begin{aligned}\mu(T_\lambda(z, +\infty) &= \mu(-1/(z-(\lambda-E)), 0)\\ &= (1-p)\,\mu(z, +\infty) + p\mu(z-\lambda, +\infty)\\ &= (z, +\infty) + p\mu(z-\lambda, z) \qquad (3.6)\end{aligned}$$

if $z > \lambda - E$.

We now use (3.2) to bound from below the sum of the first two terms in the rhs of (3.1):

$$\begin{aligned}&\int_0^1 dz\, z^{-1}\mu(0, z) - \int_1^\infty dz\, z^{-1}(-\infty, -z)\\ &= \int_0^1 dz\, z^{-1}\mu(0, z) - \int_3^\infty dz\, z^{-1}\mu(0, 1/(z-E))\\ &\quad + (1-p)\int_3^\infty dz\, z^{-1}\mu(-z, \lambda-z) - \int_1^3 dz\, z^{-1}\mu(-\infty, -z)\\ &> \int_0^1 dz\,(1+Ez)^{-1}\, E\mu(0, z) + \int_0^1 dz\, z^{-1}\mu(0, z)\\ &\quad - \int_1^3 dz\, z^{-1}\mu(-\infty, -z) + (1-p)\int_3^\infty dz\, z^{-1}\mu(-z, \lambda-z)\\ &> -a_0 + (1-p)\int_3^\infty dz\, z^{-1}\mu(-z, \lambda-z) \qquad (3.7)\end{aligned}$$

where a_0 is a constant *independent of* E *and* λ. In a similar way, we can estimate the sum of the third and fourth terms. In this case it is convenient to split the integral:

$$\int_1^\infty dz\, z^{-1}\mu(z, +\infty) = \int_3^{\lambda/2} dz\, z^{-1}\mu(z, +\infty) + \int_{3\lambda/2}^\infty dz\, z^{-1}\mu(z, +\infty) + c \qquad (3.8)$$

where

$$c = \int_1^3 dz\, z^{-1}\mu(z, +\infty) + \int_{\lambda/2}^{3\lambda/2} dz\, z^{-1}\mu(z, +\infty) < c_0 \qquad (3.9)$$

independent of E *and* λ.

Then one uses (3.3) in the first integral and (3.4) in the second. The final result is

$$\int_0^1 dz\, z^{-1}\mu(0; z) - \int_1^\infty dz\, z^{-1}\mu(z, +\infty)$$
$$> -a_1 - (1-p)\int_3^{\lambda/2} dz\, z^{-1}\mu(z, z+\lambda) \tag{3.10}$$

with a_1 *independent of E and* λ.

By combining (3.1), (3.7), and (3.10) together, we finally get

$$\gamma_\lambda(E) > -a_3 + (1-p)\int_3^{\lambda/2} dz\, z^{-1}[\mu(-z, \lambda - z) - \mu(z, z+\lambda)] \tag{3.11}$$

Thus, in order to show that $\gamma_\lambda(E)$ diverges uniformly in the energy E like const $\cdot \ln\lambda$, it is enough to prove that, uniformly in E and in $z \in [3, \lambda/2]$, the quantity

$$\mu(-z, \lambda - z) - \mu(z, z+\lambda) = \mu(-z, z) - \mu(\lambda - z, \lambda + z) \tag{3.12}$$

is greater than a positive constant $k(p)$ *independent of* λ.

To estimate (3.12), we first have to fix some notations.

We denote by $A = A(\lambda, E)$ the *stable* fixed point for the map T_λ, i.e.,

$$\begin{aligned} A &= (\lambda - E - A)^{-1} \\ &= \{\lambda - E - [(\lambda - E)^2 - 4]^{1/2}\}/2 \\ &= O(+1/\lambda) \qquad \text{as} \quad \lambda \to \infty \end{aligned} \tag{3.13}$$

Next we prove two estimates that are at the basis of the subsequent computations:

$$\mu(-z, z) > 1/2 \qquad \text{for any} \quad z > 3 \tag{3.14}$$

and

$$\mu(\lambda - z, \lambda + z) < \mu(-E - 1/(\lambda - z), -E - 1/(\lambda + z)) \tag{3.15}$$

To prove (3.14), let us apply (2.) to the set $(-\infty, -z) \cup (z, +\infty)$. We have

$$\begin{aligned} &\mu((-\infty, -z) \cup (z, +\infty)) \\ &\quad = (1-p)\,\mu(-E - 1/(\lambda - z), -E - 1/(\lambda + z)) \\ &\quad\quad + p\mu(-E - 1/z, -E + 1/z) \end{aligned} \tag{3.16}$$

which implies, for $3 < z < \lambda/2$,

$$\mu((-\infty, -z) \cup (z, +\infty)) < \mu(-E - 1/z, -E + 1/z) < \mu(-z, z) \qquad (3.17)$$

Thus (3.14) follows.

To prove (3.15), we apply (2.) to the set $(\lambda - z, \lambda + z)$:

$$\begin{aligned} \mu(\lambda - z, \lambda + z) &= (1-p)\,\mu(\lambda - E - 1/(\lambda - z), \lambda - E - 1/(\lambda + z)) \\ &\quad + p\mu(-E - 1/(\lambda - z), -E + 1/(\lambda + z)) \\ &< (1-p)\mu(\lambda - z, \lambda + z) + p\mu(-E - 1/(\lambda - z), -E - 1/(\lambda + z)) \end{aligned} \qquad (3.18)$$

and (3.15) follows.

We can proceed to the estimate of the quantity

$$\mu(-z, z) - \mu(\lambda - z, \lambda + z), \qquad 3 < z < \lambda/2 \qquad (3.19)$$

We have to distinguish between two different cases:

$$\text{case a} \qquad -E \in [-2, +2/(3\lambda)] \qquad (3.20\text{a})$$

$$\text{case b} \qquad -E \in [2/(3\lambda), 2] \qquad (3.20\text{b})$$

Notice that the function $A(E) - E$ is monotone decreasing and $A(E) = -E$ if $E = -1/\lambda$.

Case a. In this case, using (3.16), we estimate (3.19) by $\mu(-E - 1/(\lambda + z), z)$, which in turn is bounded from below by

$$\mu(-E - 1/(\lambda + z), z) > \mu(0, z) > (1-p)\,\mu(-\infty, \lambda - E - 1/z) \quad (3.21)$$

Therefore, for $-2 < -E < 2\lambda/3$ and any z, $\lambda/2 > z > 3$:

$$\mu(-E - 1/(\lambda + z), z) > (1-p)\,\mu(-z, z) > (1-p)/2 \qquad (3.22)$$

This proves, together with (3.11), that for *any* $-E \in [-2, 2\lambda/3]$ and *any* λ sufficiently large (e.g., $\lambda > 10$)

$$\gamma_\lambda(E) > [(1-p)^2/2] \ln \lambda - c \qquad (3.23)$$

with c *independent of* λ *and* E.

Case b. $2 > -E > 2/(3\lambda)$. In this case we estimate (3.20), using (3.16), by

$$\mu(-E, z) \qquad \text{if} \quad 1/\lambda > -E > 2/(3\lambda) \qquad (3.24)$$

and by

$$\mu(-z, -E-1/(\lambda-z)) \qquad \text{if} \quad 2 > -E > 1/\lambda \tag{3.25}$$

In the first case (3.24) we will prove that

$$\mu(-E, z) > (1-p)\, p^2 \mu(\lambda - z, \lambda + z) \tag{3.26}$$

Assuming (3.26), it follows that

$$\mu(-z, z) - \mu(\lambda - z, \lambda + z) > (1-p)\, p^2 \mu(\lambda - z, \lambda + z) \tag{3.27}$$

and therefore, using (3.14), we have

$$\mu(-z, z) - \mu(\lambda - z, \lambda + z) > (1-p)\, p^2/2[2 + (1-p)\, p^2] \tag{3.28}$$

Thus,

$$\gamma_\lambda(E) > (1-p)\, p^2/2[1 + (1-p)\, p^2] \ln \lambda - c \qquad \forall E \in [-1/\lambda, -2/(3\lambda)]$$

To prove (3.26), we apply (2.3) to the set $(-E, z)$ and use the fact that under the action of the map T_λ^{-1} the point $-E$ moves to the left, since $A(E) > -E$ if $-E \in [1/\lambda, 2/(3\lambda)]$. We have

$$\mu(-E, z) > (1-p)\, \mu(T_\lambda^{-1}(-E, z)) > (1-p)\, \mu(-E, \lambda - E - 1/z) \tag{3.29}$$

Using now the map T_0^{-1}, we have

$$\begin{aligned} \mu(-E, \lambda - E - 1/z) &> p\mu(T_0^{-1}(-E, \lambda - E - 1/z) \\ &= p\mu(-E + 1/E, -E + 1/(\lambda - E - 1/z)) \\ &> p^2\mu(-E + (E - 1/E)^{-1}, -E + [E + 1/(\lambda - E - 1/z)]^{-1}) \end{aligned} \tag{3.30}$$

We observe that if $1/\lambda > -E > 2/(3\lambda)$ and $\lambda/2 > z > 3$,

$$\begin{aligned} -E + (E - 1/E)^{-1} &< \lambda - z \\ -E + [E + 1/(\lambda - E - 1/z)]^{-1} &> \lambda + z \end{aligned} \tag{3.31}$$

Thus, the rhs of (3.30) is greater than $p^2\mu(\lambda - z, \lambda + z)$ and (3.26) follows.

We are left with the case (3.25), $2 > -E > A(E)$. In this case we observe that

$$-E - 1/(\lambda - z) > -1/(E + \lambda + z) \tag{3.32}$$

Therefore

$$\mu(-z, -E-1/(\lambda-z)) > \mu(-z, -1/(E+\lambda+z)) > p\mu(-E+1/z, \lambda+z)$$
$$> p\mu(\lambda-z, \lambda+z) \qquad \text{if} \quad \lambda/2 > z > 3 \qquad (3.33)$$

By the same reasons that led to (3.28), we conclude that

$$\mu(-z, z) - \mu(\lambda-z, \lambda+z) > p/[2(1+p)] \qquad (3.34)$$

i.e.,

$$\inf_{1/\lambda < E < 2} \gamma_\lambda(E) < p/[2(1+p)] \ln \lambda - c \qquad (3.35)$$

The proof is finished.

Proof of Theorem 2. To prove the theorem, we will combine the initial definition of the Liapunov exponent (1.3) together with the Thouless formula (1.7). The main idea is the following: for the special energies considered in the theorem it is possible to compute explicitly the product of the random matrices $T_j(v, E)$ associated to strings of potential v of a particular form. The result is that these strings contribute to the full product (1.3) exactly like a single matrix $T(E)$, with, however, the potential v replaced by a new, explicitly computable effective potential $\tilde{v}$. In this way one shows that the Liapunov exponent $\gamma_\lambda(E)$ is an explicit constant times the Liapunov exponent $\tilde{\gamma}_\lambda(E)$ for the effective potential $\tilde{v}$. The advantage of this procedure is that for the new potential $\tilde{v}$ the contribution of the negative singularity at $E' = E$ in the Thouless formula becomes negligible in the limit $\lambda \to +\infty$ compared with that of the positive part (1.8) and the heuristic considerations of Section 1 can be rigorously implemented. Thus, let $E = 2\cos[\pi k/(L+1)]$ with k and L as in the theorem. It is easy to see that E is an eigenvalue of the discrete Laplacian $-\Delta_{nL+n-1}$ restricted to the interval $[1, nL+n-1]$, $n \in \mathbf{N}$. This fact allows us to compute directly the $(nL+n-1)$ power of the transfer matrix T_0:

$$T_0 = \begin{pmatrix} -E, & -1 \\ 1, & 0 \end{pmatrix} \qquad (3.36)$$

We have

$$(T_0)^{nL+n-1} = \pm \begin{pmatrix} 0, & +1 \\ -1, & -E \end{pmatrix} \qquad (3.37)$$

The specific choice of the sign depends on n and L, but it is completely irrelevant for our purposes. Furthermore, since

$$T_0^{-1}=\begin{pmatrix} 0, & 1 \\ -1, & -E \end{pmatrix} \tag{3.38}$$

we have that $(T_0)^{nL+n}=\pm\mathbf{I}$.

The last algebraic identity that will be at the basis of the construction of the effective potential $\tilde{v}$ is the following:

$$\begin{pmatrix} m\lambda-E, & -1 \\ 1, & 0 \end{pmatrix}\begin{pmatrix} 0, & 1 \\ -1, & -E \end{pmatrix}\begin{pmatrix} j\lambda-E, & -1 \\ 1, & 0 \end{pmatrix}=\begin{pmatrix} (m+j)\lambda-E, & -1 \\ 1, & 0 \end{pmatrix} \tag{3.39}$$

for any $m, j \in \mathbf{N}$.

In order to construct the new potential, we first have to fix some simple notations.

We will denote by S_Q any strong of Q nearest neighbor sites and by $v\{S_Q\}$ the corresponding potential. We will also write $v\{S_Q\}=O(\lambda)$ if $v=O(\lambda)$ at any site in S_Q. By $\{S_{Q_1}, S_{Q_2},..., S_{Q_k}\}$ we will denote the string S_Q, $Q=\sum_{j=1}^{k} Q_j$, obtained by joining together in the given order the strings S_Q. Identities (3.37)–(3.39) suggest that we construct the effective potential $\tilde{v}$ out of the potential v according to the following set of rules:

a. If $v\{S_Q\}=0$ and $Q'=Q-[Q/(L+1)](L+1)<L$, then the string S_Q is replaced by the shorter string $S_{Q'}$ with $\tilde{v}\{S_{Q'}\}=0$. Here $[\cdot]$ denotes the integer part.

b. Any string S_Q such that $S_Q=\{S_{Q_1}, S_Q,..., S_{Q_m}\}$ with each S_Q having the property that: (i) $v=\lambda$ at its leftmost site and $v=0$ elsewhere; and (ii) $Q_j-1-[(Q_j-1)/(L+1)](L+1)=L$, $j=1,..., m-1$, $Q_m=1$; is replaced by a string $S_{Q'}$ of length $Q'=1$ and $v\{S_{Q'}\}=m\lambda$.

A string $S_{Q'}$ that is the result of the application of either rule a or rule b to a string S_Q will be called "renormalized."

These rules are illustrated in Fig. 2 for the simple case $L=1$, namely $E=0$.

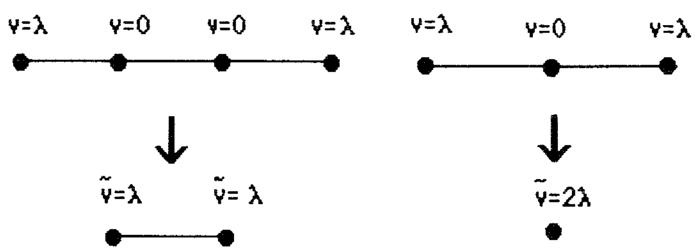

Fig. 2

Remark 1. In the above example (see Fig. 2) two sites of potential equal to λ separated by a site of zero potential give an effective potential at $E=0$ equal to 2λ and *not* of the order of λ^2 as one might naively expect. Thus, their contribution to the Liapunov exponent $\gamma_\lambda(0)$ will still be $\log \lambda$ and *not* $2 \log \lambda$. This explain why one gets values of the Liapunov exponent smaller than $(1-p)\log \lambda$.

Remark 2. The construction of the effective potential $\tilde{v}$ is done directly on the infinite configuration $v=\{v(j)\}_{j\in\mathbf{Z}}$. It is clear that with the exception of a set of measure zero, the configuration $\tilde{v}$ will be an infinite sequence of random variables taking values 0, λ, 2λ,... labeled by numbers in $\mathbf{Z}$. The random variables $\tilde{v}(j)$ and $\tilde{v}(m)$ are independent provided that $|m-j|>L$. Note that by construction the maximum length of a string of $\tilde{v}=0$ is $L-1$.

In order to fix the origin of the new configuration $\tilde{v}$, we adopt the following convention. If the site $x=0$ belongs to a string S_Q that can be renormalized using rule a, then: (i) if $Q=n(L+1)$ for some n, the new origin is set at the site immediately to the right of S_Q; (ii) if $Q>Q/(L+1)$, the new origin will be the leftmost site of the renormalized string $S_{Q'}$. If the site $x=0$ belongs to a string S_Q that can be renormalized using rule b, or if it does not belong to a renormalizable string [i.e., $v(0)=\lambda$ and neither to the left nor to the right is there a string S_Q, $Q=n(L+1)+L$, for some n, with $v\{S_Q\}=0$], then the new origin will coincide with the old one.

It is clear from rules a and b that in the construction of the effective potential there is a kind of decimation procedure. However, the number of variables $\tilde{v}$ that are obtained from n variables v is roughly speaking proportional to n. More precisely, let

$$N^-(n, v) = \#\{j>0;\ \tilde{v}(j) \text{ depends only on } v(k),\ k\in[1,..., n]\}$$

and let

$$\begin{aligned} N^+(n, v) = N^-(n, v) + \#\{j>0;\ &\tilde{v}(j) \text{ is obtained by} \\ &\text{applying rule a or rule b to a string} \\ &S_Q \text{ such that } S_Q \cap [1,..., n] \neq \varnothing, \\ &\text{but } S_Q \not\subset [1,..., n]\} \end{aligned}$$

Clearly $N^+(n, v) - N^-(n, v) < 2(L-1)$ by construction.

Furthermore, using the ergodic theorem, one easily proves that

$$\lim_{n\to\infty} N^-(n, v)/n = g(p, L) \tag{3.40}$$

with probability 1. The specific value of $g(p, L)$ would be easy to compute, but for reasons that will be clear later, we do not need it.

Using Remark 2 together with (1.3) and (3.37)–(3.39), we get finally the following basic identity

$$\gamma_\lambda(E) = \tilde{\gamma}_\lambda(E)\, g(p, L) \tag{3.41}$$

where $\tilde{\gamma}_\lambda(E)$ denotes the Liapunov exponent for $\tilde{v}$.

Let now $d\tilde{N}_\lambda(E')$ be the i.d.s. measure for the potential $\tilde{v}$ [see (1.5)].

Since the maximum length of a string S_Q with $\tilde{v}\{S_Q\} = 0$ is $L-1$ and since the energy under consideration is *not* an eigenvalue of $-\Delta_Q$ for $Q < L$ (we are using here the fact that $L+1$ and k are mutually prime), we have that for λ so large that $1/\lambda \ll \delta/2$, $\delta \equiv \min_{Q<L} \operatorname{dist}(E, \operatorname{spec}(-\Delta_Q))$,

$$\tilde{N}_\lambda(E+\delta/2) - \tilde{N}_\lambda(E-\delta/2) = 0 \tag{3.42}$$

Therefore, the first integral in the Thouless formula

$$\tilde{\gamma}_\lambda(E) = \int_{-2}^{+2} d\tilde{N}_\lambda(E') \ln |E-E'| + \int_{\lambda-2}^{\infty} d\tilde{N}_\lambda(E') \ln |E-E'| \tag{3.43}$$

will stay bounded uniformly in λ.

Next we compute the asymptotics as $\lambda \to \infty$ of the second term in (3.43). Since the effective potential $\tilde{v}$ takes values that are integers multiples of λ, the second integral can be written as

$$\int_{\lambda-2}^{\infty} d\tilde{N}_\lambda(E') \ln |E-E'| = \sum_{j=1}^{\infty} \int_{j\lambda-2}^{j\lambda+2} d\tilde{N}_\lambda(E') \ln |E-E'| \tag{3.44}$$

It is easy to see that

$$\sum_{j=1}^{\infty} \ln |j| \int_{j\lambda-2}^{j\lambda+2} d\tilde{N}_\lambda(E') < C \quad \text{uniformly in } \lambda \tag{3.45}$$

In fact,

$$\tilde{N}_\lambda(j\lambda+2) - \tilde{N}_\lambda(j\lambda-2) \approx P\{\tilde{v}(0) = j\lambda\}$$

and this last probability obviously decays exponentially fast in j.

Therefore we can conclude that

$$\begin{aligned} &\lim_{\lambda\to\infty} \left[\int_{\lambda-2}^{\infty} d\tilde{N}_\lambda(E') \ln |E-E'|\right] \Big/ \ln \lambda \\ &= \sum_{j=1}^{\infty} \int_{j\lambda-2}^{j\lambda+2} d\tilde{N}_\lambda(E') \\ &= 1 - \tilde{N}_\lambda(\lambda-2) = 1 + \tilde{N}_\lambda(2) \end{aligned} \tag{3.46}$$

that is

$$\lim_{\lambda \to \infty} \gamma_\lambda(E)/\log \lambda = [1 - \tilde{N}_\lambda(2)]\ g(p, L) \tag{3.47}$$

To compute the rhs of (3.47), we observe that, using the ergodic theorem, one has

$$\begin{aligned}
&\lim_{n \to \infty} 1/n\{\#(0 < j < n; v(j) = \lambda) \\
&\quad - \#(\text{strings } S_Q \subset [0, n]; Q = j(L+1) + L \text{ for some } j \text{ and } v\{S_Q\} = 0)\} \\
&= \lim_{n \to \infty} N^-(n, v)/nN^-(n, v)\{\#(0 < j < N^-(n, v); \tilde{v}(j) > 0)\} \\
&= g(p, L)[1 - \tilde{N}_\lambda(2)]
\end{aligned} \tag{3.48}$$

On the other hand, the left-hand side of (3.48) is equal to

$$-p - (1-p)^2 \left(\sum_{j=0}^{\infty} p^{j(L+1)+L} \right) = 1 - p - (1-p)^2 p^L/(1 - p^{L+1}) \tag{3.49}$$

and the proof is finished.

4. NUMERICAL RESULTS

We conclude this work by presenting some numerical results on γ_λ. They have been obtained by computing by the Monte Carlo method the Liapunov exponent of a string of 10^4 sites. The probability p was taken equal to 1/2 and the coupling constant λ equal to 100. In Fig. 3 the energy step is 0.01, while in Fig. 4 it is taken equal to 0.0005. The values for $\gamma_\lambda(0)$ and $\gamma_\lambda(1)$ that we obtain are 0.338 ln λ and 0.440 ln λ, respectively, in good agreement with the predicted values

$$(1-p)/(1+p) \log \lambda = 1/3 \ln \lambda$$

$$(1-p^2)/(1+p^2+p) \ln \lambda = 3/7 \ln \lambda$$

It is clear from the figures that the minimum of the Liapunov exponent is attained at $E = -2/\lambda$, where we obtain

$$\gamma_\lambda(-2/\lambda) \approx 0.22 \log \lambda < 1/2(1-p) \log \lambda$$

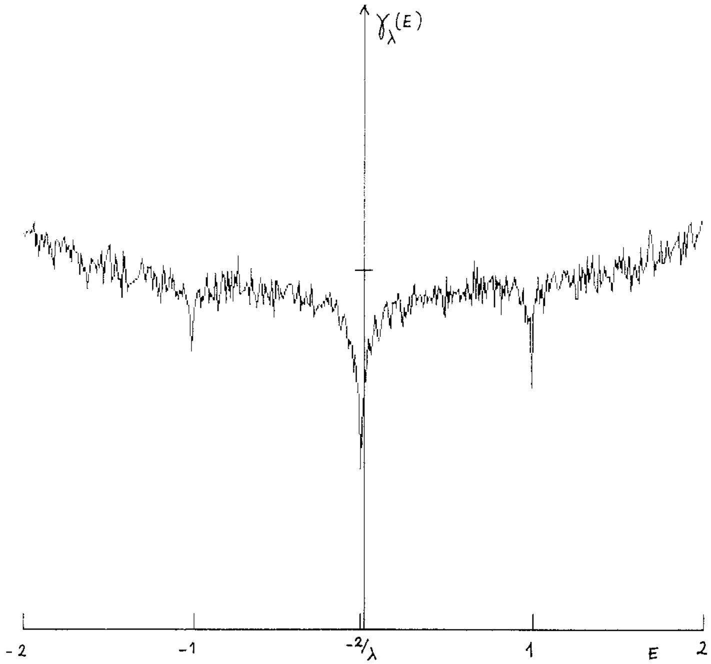

Fig. 3. Graph of the Liapunov exponent for $p = 0.5$, $\lambda = 100$.

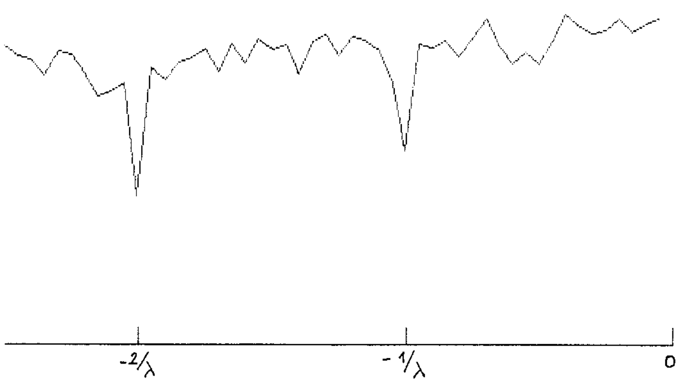

Fig. 4. Graph of the Liapunov exponent around $E = -2/\lambda$ for $p = 0.5$, $\lambda = 100$.

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